Technical Appendix to

"Optimal Monetary and Fiscal Policy Rules, Welfare Gains and Exogenous Shocks in an Economy with Default Risk"

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A Derivation of Secon-order Approximated Utility Function

A.1 Derivation of Second-order approximated utility function with Default Risk

Government solvency condition stemming from Eq.(16) in the text is given by:

$$W_t\left(1-\delta_t\right)=C_t^{-1}SP_t+\beta \mathrm{E}_t R_t^R\left(C_{t+1}^{-1}SP_{t+1}\right)+\beta^2 \mathrm{E}_t R_t^R R_{t+1}^R\left(C_{t+2}^{-1}SP_{t+2}\right)+\cdots$$

with $W_t \equiv C_t^{-1}R_{t-1}^G B_{t-1}\Pi_t^{-1}$, which can be rewritten as:

$$\begin{aligned}W_t\left(R^R\right)^{-1}=&\left[\left(1-\delta_t\right)R^R\right]^{-1}C_t^{-1}SP_t+\beta\left[\left(1-\delta_t\right)R^R\right]^{-1}R_t^R C_{t+1}^{-1}SP_{t+1}\\&+\beta^2\left[\left(1-\delta_t\right)R^R\right]^{-1}R_t^R R_{t+1}^R C_{t+2}^{-1}SP_{t+2}+\cdots\end{aligned}, \quad (A.1)$$

with. Notice that $R^R=R^S=\left(1-\delta\right)^{-1}$.

Forwarding Eq.(A.1) one period yields:

$$\begin{aligned}W_{t+1}\left(R^R\right)^{-1}=&\left[\left(1-\delta_{t+1}\right)R^R\right]^{-1}C_{t+1}^{-1}SP_{t+1}+\beta\left[\left(1-\delta_{t+1}\right)R^R\right]^{-1}R_{t+1}^R C_{t+2}^{-1}SP_{t+2}\\&+\beta^2\left[\left(1-\delta_{t+1}\right)R^R\right]^{-1}R_{t+1}^R R_{t+2}^R C_{t+3}^{-1}SP_{t+3}+\cdots\end{aligned}.$$

Plugging the previous expression into Eq.(A.1) yields:

$$W_t\left(R^R\right)^{-1}=\left[\left(1-\delta_t\right)R^R\right]^{-1}C_t^{-1}SP_t+\beta\left(R^R\right)^{-1}R_t^R\left(1-\delta_{t+1}\right)\left(1-\delta_t\right)^{-1}W_{t+1}.$$

In the period zero, the previous expression can be rewritten as:

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$$W_0\left(R^R\right)^{-1}=\left[(1-\delta_0)R^R\right]^{-1}C_0^{-1}SP_0+\beta\left(R^R\right)^{-1}R_0^R(1-\delta_1)(1-\delta_0)^{-1}W_1.$$

Note that δ_0 is predetermined and is constant. Then,

$$W_0\left(R^R\right)^{-1}=\widetilde{SP}_0+\beta\left(R^R\right)^{-1}R_0^R(1-\delta_1)(1-\delta)^{-1}W_1, \text{ (A.2)}$$

with $\widetilde{SP}_0\equiv C_0^{-1}SP_0$.

Let define $LW_0\equiv\widetilde{SP}_0+\beta\left(R^R\right)^{-1}R_0^R(1-\delta_1)(1-\delta)^{-1}W_1$.

Taking logarithm the previous expression yields:

$$\begin{aligned}\ln LW_0&=\ln LW_0-\ln LW+\ln LW\\&=\ln\left(\frac{LW_0}{LW}\right)+\ln LW\end{aligned}.$$

First order approximation of Eq.(A.2) yields:

$$\begin{aligned}W_0\left(R^R\right)^{-1}&=W\left(R^R\right)^{-1}+f(W_0)_{\widetilde{SP}}\left(\widetilde{SP}_0-\widetilde{SP}\right)+f(W_0)_{R^R}\left(R_0^R-R^R\right)+f(W_0)_\delta(\delta_0-\delta)\\&\quad+f(W_0)_W(W_1-W)\\&=W\left(R^R\right)^{-1}+\widetilde{SP}\widetilde{sp_0}+\beta\left(R^R\right)^{-1}WR^R\hat{r}_0^R-\beta(1-\delta)^{-1}W\delta\hat{\delta}_1+\beta Ww_1\end{aligned}.$$

Dividing both sides of the previous expression yields:

$$\frac{W_0}{W}\left(R^R\right)^{-1}=\left(R^R\right)^{-1}+\frac{\widetilde{SP}}{W}\widetilde{sp_0}+\beta\left(R^R\right)^{-1}R^R\hat{r}_0^R-\beta(1-\delta)^{-1}\delta\hat{\delta}_1+\beta w_1,$$

which can be rewritten as:

$$\ln\left(\frac{LW_0}{LW}\right)=(1-\delta-\beta)\widetilde{sp}_0+\beta\hat{r}_0^R-\beta\frac{\delta}{1-\delta}\hat{\delta}_1+\beta w_1, \text{ (A.3)}$$

where we use $\left(R^R\right)^{-1}\frac{W_0}{W}-1=\ln\left(\frac{LW_0}{LW}\right)$ and $\frac{\widetilde{SP}}{W}=1-\delta-\beta$.

Taking logarithm of the LHS of Eq.(A.3) yields:

$$\begin{aligned}\ln W_0-\ln W-\ln R^R&=\ln\left(\frac{W_0}{W}\right)-\ln R^R\\&=w_0-\ln R^R\end{aligned}. \text{ (A.4)}$$

Combining Eq.(3) with Eq.(4) yields:

$$w_0=(1-\delta-\beta)\widetilde{sp}_0+\beta\hat{r}_0^R-\beta\frac{\delta}{1-\delta}\hat{\delta}_1+\beta w_1.$$

Plugging $\hat{\delta}_t=\frac{1-\delta}{\delta}\hat{r}_{t-1}^S$ into the previous expression yields:

$$w_0 = (1-\delta-\beta)\widetilde{sp}_0 + \beta\hat{r}_0^R - \beta\hat{r}_0^S + \beta w_1 \text{. (A.5)}$$

Integrating Eq.(A.5) forward yields:

$$\varpi = (1-\delta-\beta)\sum_{t=0}^{\infty} \mathrm{E}_0\left(\widetilde{sp}_t + \frac{\beta}{1-\beta-\delta}\hat{r}_0^R - \frac{\beta}{1-\beta-\delta}\hat{r}_0^S\right) \text{, A.6)}$$

with $\varpi \equiv w_0$ where we use:

$$w_0 = (1-\delta-\beta)\left(\widetilde{sp}_0 + \frac{\beta}{1-\delta-\beta}\hat{r}_0^R - \frac{\beta}{1-\delta-\beta}\hat{r}_0^S + \frac{\beta}{1-\delta-\beta}w_1\right),$$

$$w_1 = (1-\delta-\beta)\left(\widetilde{sp}_1 + \frac{\beta}{1-\delta-\beta}\hat{r}_1^R - \frac{\beta}{1-\delta-\beta}\hat{r}_1^S + \frac{\beta}{1-\delta-\beta}w_2\right),$$

$$w_2 = (1-\delta-\beta)\left(\widetilde{sp}_2 + \frac{\beta}{1-\delta-\beta}\hat{r}_2^R - \frac{\beta}{1-\delta-\beta}\hat{r}_2^S + \frac{\beta}{1-\delta-\beta}w_3\right),$$

….

Second order approximation of $\widetilde{SP}_t = C_t^{-1}SP_t$ is given by:

$$\widetilde{sp}_t = -c_t + sp_t + \frac{1}{2}c_t^2 - c_t sp_t + o\left(\|\xi\|^3\right) \text{, (A.7)}$$

where we use $\frac{C_t^{-1}SP_t}{C^{-1}SP} - 1 = \widetilde{sp}_t$.

Plugging Eq.(A.7)into Eq.(6) yields:

$$\varpi = (1-\delta-\beta)\sum_{t=0}^{\infty} \mathrm{E}_0\left[-c_t + sp_t + \frac{1}{2}c_t^2 - c_t sp_t + \frac{\beta}{1-\beta-\delta}\left(\hat{r}_t^R - \hat{r}_t^S\right)\right] \text{. (A.9)}$$

Second-order approximation $SP_t \equiv \tau_t Y_t - G_t$ is given by:

$$\begin{aligned} SP_t = {} & SP + SP_\tau(\tau_t - \tau) + SP_Y(Y_t - Y) + SP_G(G_t - G) + \frac{1}{2}SP_{\tau\tau}(\tau_t - \tau)^2 + \frac{1}{2}SP_{YY}(Y_t - Y)^2 \\ & + \frac{1}{2}SP_{\tau Y}(\tau_t - \tau)(Y_t - Y) + \text{s.o.t.i.p.} + o\left(\|\xi\|^3\right) \end{aligned},$$

which can be rewritten as:

$$sp_t = (1+\omega_g)(\hat{\tau}_t + y_t) - \omega_g g_t + \frac{1+\omega_g}{2}\left(\frac{SP_{\tau\tau}\tau}{SP_\tau}\hat{\tau}_t^2 + \frac{SP_{YY}Y}{SP_Y}y_t^2 + 2\hat{\tau}_t y_t\right) + \text{s.o.t.i.p.} + o\left(\|\xi\|^3\right),$$

with $\omega_g \equiv \frac{G}{SP}$.

First-order approximation of $SP_t \equiv \tau_t Y_t - G_t$ is given by:

$$sp_t = \frac{\tau Y}{SP}\hat{\tau}_t + \frac{\tau Y}{SP}y_t - \frac{G}{SP}g_t + o\left(\|\xi\|^2\right).$$

Here, $\frac{\tau Y}{SP}$ is the elasticity of tax rate or output to primary balance. Thus, this equality implies that the elasticity of tax rate to primary balance is equal to the elasticity of output to primary balance. In addition, one percent increase in tax rate increases primary balance one percent and one percent increase in output increases primary balance one percent. This fact implies that $\frac{SP_{\tau\tau}\tau}{SP_{\tau}} = \frac{SP_{YY}Y}{SP_{Y}} = 1$. Then, we have:

$$\begin{aligned} sp_t = & \left(1+\omega_g\right)\hat{\tau}_t + \left(1+\omega_g\right)y_t + \frac{1+\omega_g}{2}\hat{\tau}_t^2 + \frac{1+\omega_g}{2}y_t^2 + \left(1+\omega_g\right)\hat{\tau}_t y_t \\ & -\omega_g g_t + \text{s.o.t.i.p.} + o\left(\|\xi\|^3\right) \end{aligned} . \text{(A.10)}$$

Plugging Eq.(A.10) into Eq.(A.9) yields:

$$\varpi = \left(1-\delta-\beta\right)\sum_{t=0}^{\infty}\mathrm{E}_0 \begin{bmatrix} -c_t + \left(1+\omega_g\right)\hat{\tau}_t + \left(1+\omega_g\right)y_t \\ +\frac{\beta}{1-\beta-\delta}\hat{r}_t^R - \frac{\beta}{1-\beta-\delta}\hat{r}_t^S + \frac{1}{2}c_t^2 \\ +\frac{1+\omega_g}{2}\hat{\tau}_t^2 + \frac{1+\omega_g}{2}y_t^2 - \left(1+\omega_g\right)c_t\hat{\tau}_t \\ -\left(1+\omega_g\right)c_t y_t + \omega_g c_t g_t + \left(1+\omega_g\right)\hat{\tau}_t y_t \end{bmatrix} + \text{t.i.p} + \left(\|\xi\|^3\right). \text{(A.11)}$$

Second-order approximation of $\Psi\left(R_t^R\right)$ is given by:

$$\begin{aligned} \Psi\left(R_t^R\right) &= \Psi\left(R^R\right) + \Psi_{R^R}\left(R^R\right)\left(R_t^R - R^R\right) + \frac{1}{2}\Psi_{R^R R^R}\left(R^R\right)\left(R_t^R - R^R\right)^2 + o\left(\|\xi\|^3\right) \\ &= \Psi\left(R^R\right) + \Psi_{R^R}\left(R^R\right)R^R\left[\hat{r}_t^R + \frac{1}{2}\left(\hat{r}_t^R\right)^2\right] + \frac{1}{2}\Psi_{R^R R^R}\left(R^R\right)\left(R^R\right)^2\left(\hat{r}_t^R\right)^2 + o\left(\|\xi\|^3\right) \end{aligned} .$$

Let define $\Psi\left(R_t^R\right) \equiv \left(R_t^R\right)^{\left(\frac{1}{\gamma-1}\right)^2}$. Then we have $\Psi_{R^R}\left(R^R\right) = \left(\frac{1}{\gamma-1}\right)^2\left(R^R\right)^{\left(\frac{1}{\gamma-1}\right)^2-1}$ and $\Psi_{R^R R^R}\left(R^R\right) = \left(\frac{1}{\gamma-1}\right)^2\left[\left(\frac{1}{\gamma-1}\right)^2 - 1\right]\left(R^R\right)^{\left(\frac{1}{\gamma-1}\right)^2-2}$. Plugging those derivatives into the previous expression yields:

$$\Psi\left(R_t^R\right)=\Psi\left(R^R\right)+\left(\frac{1}{\gamma-1}\right)^2\left(R^R\right)^{\left(\frac{1}{\gamma-1}\right)^2-1}R^R\left[\hat{r}_t^R+\frac{1}{2}\left(\hat{r}_t^R\right)^2\right]$$
$$+\frac{1}{2}\left(\frac{1}{\gamma-1}\right)^2\left[\left(\frac{1}{\gamma-1}\right)^2-1\right]\left(R^R\right)^{\left(\frac{1}{\gamma-1}\right)^2-2}\left(R^R\right)^2\left(\hat{r}_t^R\right)^2+o\left(\|\xi\|^3\right).$$
$$=\Psi\left(R^R\right)+\left(\frac{1}{\gamma-1}\right)^2\left(R^R\right)^{\left(\frac{1}{\gamma-1}\right)^2}\left[\hat{r}_t^R+\frac{1}{2}\left(\hat{r}_t^R\right)^2\right]$$
$$+\frac{1}{2}\left(\frac{1}{\gamma-1}\right)^2\left[\left(\frac{1}{\gamma-1}\right)^2-1\right]\left(R^R\right)^{\left(\frac{1}{\gamma-1}\right)^2}\left(\hat{r}_t^R\right)^2+o\left(\|\xi\|^3\right)$$

Subtracting $\Psi\left(R^R\right)$ from both sides of the previous expression and dividing both sides of the previous expression by that yields:

$$\frac{\Psi\left(R_t^R\right)-\Psi\left(R^R\right)}{\Psi\left(R^R\right)}=\left(\frac{1}{\gamma-1}\right)^2\hat{r}_t^R+\frac{1}{2}\left(\frac{1}{\gamma-1}\right)^2\left\{1+\left[\left(\frac{1}{\gamma-1}\right)^2-1\right]\right\}\left(\hat{r}_t^R\right)^2+o\left(\|\xi\|^3\right).$$
$$=\left(\frac{1}{\gamma-1}\right)^2\hat{r}_t^R+\frac{1}{2}\left(\frac{1}{\gamma-1}\right)^2\left(\frac{1}{\gamma-1}\right)^2\left(\hat{r}_t^R\right)^2+o\left(\|\xi\|^3\right)$$

Plugging $\frac{\Psi\left(R_t^R\right)-\Psi\left(R^R\right)}{\Psi\left(R^R\right)}=\left(\frac{1}{\gamma-1}\right)^2\hat{r}_t^R+o\left(\|\xi\|^2\right)$ into the LHS of the previous expression yields:

$$\left(\frac{1}{\gamma-1}\right)^2\hat{r}_t^R=\left(\frac{1}{\gamma-1}\right)^2\hat{r}_t^R+\frac{1}{2}\left[\left(\frac{1}{\gamma-1}\right)^2\right]^2\left(\hat{r}_t^R\right)^2+o\left(\|\xi\|^3\right).$$

Subtracting $\left(\frac{1}{\gamma-1}\right)^2\hat{r}_t^S$ from both sides of the previous expression yields:

$$\left(\frac{1}{\gamma-1}\right)^2\left(\hat{r}_t^R-\hat{r}_t^S\right)=\left(\frac{1}{\gamma-1}\right)^2\left(\hat{r}_t^R-\hat{r}_t^S\right)+\frac{1}{2}\left[\left(\frac{1}{\gamma-1}\right)^2\right]^2\left(\hat{r}_t^R\right)^2+o\left(\|\xi\|^3\right). \text{ (A.12)}$$

Second-order approximation of R_t^R yields:

$$R_t^R = \frac{R^H}{R^G} + \left(\frac{R^H}{R^G}\right)_{R^H}\left(R_t^H - R^H\right) + \left(\frac{R^H}{R^G}\right)_{R^G}\left(R_t^G - R^G\right) + \frac{1}{2}\left(\frac{R^H}{R^G}\right)_{R^G R^G}\left(R_t^G - R^G\right)^2$$

$$+\left(\frac{R^H}{R^G}\right)_{R^H R^G}\left(R_t^H - R^H\right)\left(R_t^G - R^G\right) + o\left(\|\xi\|^3\right)$$

$$= R^R + \frac{1}{R^G}R^H\left[\hat{r}_t^H + \frac{1}{2}\left(\hat{r}_t^H\right)^2\right] - \left(R_t^G\right)^{-2}R^H R^G\left[\hat{r}_t^G + \frac{1}{2}\left(\hat{r}_t^G\right)^2\right] + \left(R_t^G\right)^{-3}R^H\left(R_t^G\right)^2\left(\hat{r}_t^G\right)^2$$

$$-\left(R_t^G\right)^{-2}R^H R_t^G \hat{r}_t^H \hat{r}_t^G + o\left(\|\xi\|^3\right)$$

Subtracting R^R from both sides of the previous expression and dividing both sides of the previous expression by that yields:

$$\frac{R_t^R - R^R}{R^R} = \frac{R^G}{R^H}\left\{\frac{R^H}{R^G}\left[\hat{r}_t^H + \frac{1}{2}\left(\hat{r}_t^H\right)^2\right] - \frac{R^H}{R^G}\left[\hat{r}_t^G + \frac{1}{2}\left(\hat{r}_t^G\right)^2\right] + \frac{R^H}{R^G}\left(\hat{r}_t^G\right)^2 - \frac{R^H}{R^G}\hat{r}_t^H\hat{r}_t^G\right\} + o\left(\|\xi\|^3\right),$$

which can be rewritten as:

$$\hat{r}_t^R = \hat{r}_t^S + \phi sp_t + \frac{1}{2}\hat{r}_t^R + o\left(\|\xi\|^3\right),$$

where we use $\frac{R_t^R - R^R}{R^R} = \hat{r}_t^R + o\left(\|\xi\|^2\right)$, $sp_t = \frac{1}{\phi}\left(\hat{r}_t - \hat{r}_t^G\right)$, $\hat{r}_t^S = \hat{r}_t^H - \hat{r}_t$ and $\hat{r}_t^R = \hat{r}_t^H - \hat{r}_t^G$. Subtracting $\hat{r}_t^S$ from the both sides of the previous expression yields:

$$\hat{r}_t^R - \hat{r}_t^S = \phi sp_t + \frac{1}{2}\hat{r}_t^R + o\left(\|\xi\|^3\right).$$

Plugging the previous expression into Eq.(A.12)

$$\left(\frac{1}{\gamma-1}\right)^2\left(\hat{r}_t^R - \hat{r}_t^S\right) = \left(\frac{1}{\gamma-1}\right)^2\left(\phi sp_t + \frac{1}{2}\hat{r}_t^R\right) + \frac{1}{2}\left[\left(\frac{1}{\gamma-1}\right)^2\right]^2\left(\hat{r}_t^R\right)^2 + o\left(\|\xi\|^3\right)$$

$$= \left(\frac{1}{\gamma-1}\right)^2\phi sp_t + \left(\frac{1}{\gamma-1}\right)^2\frac{1+(\gamma-1)^2}{2(\gamma-1)^2}\left(\hat{r}_t^R\right)^2 + o\left(\|\xi\|^3\right).$$

Adding $\left(\frac{1}{\gamma-1}\right)^2\hat{r}_t^S$ on both sides of the previous expression yields:

$$\left(\frac{1}{\gamma-1}\right)^2\hat{r}_t^R = \left(\frac{1}{\gamma-1}\right)^2\left[\hat{r}_t^S + \phi sp_t + \frac{1+(\gamma-1)^2}{2(\gamma-1)^2}\left(\hat{r}_t^R\right)^2\right] + o\left(\|\xi\|^3\right).$$

Plugging the previous expression into $\frac{\beta}{1-\beta-\delta}\hat{r}_t^R = \frac{\beta(\gamma-1)^2}{(1-\beta-\delta)(\gamma-1)^2}\hat{r}_t^R$ yields:

$$
\begin{aligned}
\frac{\beta}{1-\beta-\delta}\hat{r}_t^R &= \frac{\beta(\gamma-1)^2}{(1-\beta-\delta)}\left(\frac{1}{\gamma-1}\right)^2\hat{r}_t^R \\
&= \frac{\beta(\gamma-1)^2}{(1-\beta-\delta)}\left(\frac{1}{\gamma-1}\right)^2\left[\hat{r}_t^S+\phi sp_t+\frac{1+(\gamma-1)^2}{2(\gamma-1)^2}\left(\hat{r}_t^R\right)^2\right]+o\left(\|\xi\|^3\right). \quad \text{(A.13)} \\
&= \frac{\beta}{1-\beta-\delta}\left[\hat{r}_t^S+\phi sp_t+\frac{1+(\gamma-1)^2}{2(\gamma-1)^2}\left(\hat{r}_t^R\right)^2\right]+o\left(\|\xi\|^3\right)
\end{aligned}
$$

Plugging Eq.(A.13) into Eq.(A.11) yields:

$$
\varpi=(1-\delta-\beta)\sum_{t=0}^{\infty}\mathrm{E}_0\begin{bmatrix}
-c_t+(1+\omega_g)\hat{\tau}_t+(1+\omega_g)y_t \\
+\frac{\beta}{1-\beta-\delta}\left[\hat{r}_t^S+\phi sp_t+\frac{1+(\gamma-1)^2}{2(\gamma-1)^2}\left(\hat{r}_t^R\right)^2\right] \\
-\frac{\beta}{1-\beta-\delta}\hat{r}_t^S+\frac{1}{2}c_t^2 \\
+\frac{1+\omega_g}{2}\hat{\tau}_t^2+\frac{1+\omega_g}{2}y_t^2-(1+\omega_g)c_t\hat{\tau}_t \\
-(1+\omega_g)c_ty_t+\omega_gc_tg_t+(1+\omega_g)\hat{\tau}_ty_t
\end{bmatrix}+\text{t.i.p}+\left(\|\xi\|^3\right)
$$

$$
=(1-\delta-\beta)\sum_{t=0}^{\infty}\mathrm{E}_0\begin{bmatrix}
-c_t+(1+\omega_g)\hat{\tau}_t+(1+\omega_g)y_t+\frac{\phi\beta}{1-\beta-\delta}sp_t \\
+\frac{\beta\left[1+(\gamma-1)^2\right]}{2(1-\beta-\delta)(\gamma-1)^2}\left(\hat{r}_t^R\right)^2+\frac{1}{2}c_t^2+\frac{1+\omega_g}{2}\hat{\tau}_t^2 \\
+\frac{1+\omega_g}{2}y_t^2-(1+\omega_g)c_t\hat{\tau}_t-(1+\omega_g)c_ty_t \\
+\omega_gc_tg_t+(1+\omega_g)\hat{\tau}_ty_t
\end{bmatrix}+\text{t.i.p}+\left(\|\xi\|^3\right).
$$

Plugging Eq.(A.10) into the previous expression yields:

$$\varpi = (1-\delta-\beta)\sum_{t=0}^{\infty} \mathrm{E}_0 \left\{ \begin{array}{l} -c_t + (1+\omega_g)\hat{\tau}_t + (1+\omega_g) y_t \\ + \dfrac{\phi\beta}{1-\beta-\delta}\left[\begin{array}{l} (1+\omega_g)\hat{\tau}_t + (1+\omega_g) y_t \\ +(1+\omega_g)\hat{\tau}_t^2 + (1+\omega_g) y_t^2 \\ +(1+\omega_g)\hat{\tau}_t y_t - \omega_g g_t \end{array} \right] \\ + \dfrac{\beta\left[1+(\gamma-1)^2\right]}{2(1-\beta-\delta)(\gamma-1)^2}\left(\hat{r}_t^R\right)^2 + \dfrac{1}{2}c_t^2 \\ + \dfrac{1+\omega_g}{2}\hat{\tau}_t^2 + \dfrac{1+\omega_g}{2} y_t^2 - (1+\omega_g) c_t \hat{\tau}_t \\ -(1+\omega_g) c_t y_t + \omega_g c_t g_t + (1+\omega_g)\hat{\tau}_t y_t \end{array} \right\} + \text{t.i.p} + \left(\|\xi\|^3\right)$$

$$= (1-\delta-\beta)\sum_{t=0}^{\infty} \mathrm{E}_0 \left\{ \begin{array}{l} -c_t + \dfrac{(1+\omega_g)\omega_\phi}{1-\beta-\delta}\hat{\tau}_t + \dfrac{(1+\omega_g)\omega_\phi}{1-\beta-\delta} y_t \\ + \dfrac{1}{2}c_t^2 + \dfrac{(1+\omega_g)\omega_\phi}{2(1-\beta-\delta)}\hat{\tau}_t^2 + \dfrac{(1+\omega_g)\omega_\phi}{2(1-\beta-\delta)} y_t^2 \\ + \dfrac{\beta\left[1+(\gamma-1)^2\right]}{2(1-\beta-\delta)(\gamma-1)^2}\left(\hat{r}_t^R\right)^2 \\ -(1+\omega_g) c_t \hat{\tau}_t - (1+\omega_g) c_t y_t + \omega_g c_t g_t \\ + \dfrac{(1+\omega_g)\omega_\phi}{1-\beta-\delta}\hat{\tau}_t y_t \end{array} \right\} + \text{t.i.p} + \left(\|\xi\|^3\right), \quad (A.14)$$

with $\omega_\phi \equiv 1-\beta(1-\phi)-\delta$.

Second-order approximation of the market clearing $Y_t = C_t + G_t$ is given by:

$$Y(C_t, G_t) = Y + Y_C C\left(c_t + \frac{1}{2}c_t^2\right) + Y_G G g_t + \frac{1}{2}Y_{CC}C^2 c_t^2 + Y_{CG} C G c_t g_t + \text{s.o.t.i.p} + o\left(\|\xi\|^3\right),$$

which can be rewritten as:

$$y_t = \sigma_C c_t + \sigma_G g_t + \frac{\sigma_C \sigma_G}{2} c_t^2 - \sigma_G c_t g_t + \text{s.o.t.i.p} + o\left(\|\xi\|^3\right),$$

where we use $Y_C \dfrac{C}{Y} = -\dfrac{Y_{CC} C}{Y_C} = \sigma_C$, $Y_G \dfrac{G}{Y} = \sigma_G$ and $Y_{CG} = -C^{-1}$.

Iterating the previous expression forward yields:

$$0=\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left(-y_t+\sigma_C c_t+\sigma_G g_t+\frac{\sigma_C\sigma_G}{2}c_t^2-\sigma_G c_t g_t\right)+\text{t.i.p}+o\left(\|\xi\|^3\right). \text{ (A.15)}$$

Second-order approximation of Eq.(23) in the text is given by:

$$\nu=\kappa\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left[\widetilde{kk}_{t,t}-\widetilde{ff}_{t,t}+\frac{1}{2}\left(\widetilde{kk}_{t,t}^2-\widetilde{ff}_{t,t}^2\right)\right]+\text{t.i.p}+o\left(\|\xi\|^3\right), \text{ (A.16)}$$

with:

$$\widetilde{kk}_{t,t}\equiv-\sigma_G c_t+\sigma_G g_t+mc_t, \text{ (A.17)}$$

$$\widetilde{ff}_{t,t}\equiv-\sigma_G c_t+\sigma_G g_t. \text{ (A.18)}$$

Second-order approximation of Eq.(25) in the text yields:

$$\begin{aligned}mc_t=&\,c_t+\varphi y_t+\frac{\tau}{1-\tau}\hat{\tau}_t-(1+\varphi)a_t+\frac{1}{2}c_t^2+\frac{\varphi}{2}y_t^2\\&+\frac{\tau(1+2\varepsilon_\tau)}{2(1-\tau)}\hat{\tau}_t^2+\varphi c_t y_t+\varepsilon_C c_t\hat{\tau}_t-(1+\varphi)c_t a_t\\&+\varphi(\varphi-1)y_t a_t+\varphi\varepsilon_N y_t\hat{\tau}_t-(\varepsilon_A+\varphi)\hat{\tau}_t a_t+\text{s.o.t.i.p.}+o\left(\|\xi\|^3\right)\end{aligned}, \text{ (A.19)}$$

with $\varepsilon_\tau\equiv\frac{MC_{\tau\tau}\tau}{MC_\tau}$, $\varepsilon_C\equiv\frac{MC_{C\tau}\tau}{MC_C}$, $\varepsilon_N\equiv\frac{MC_{N\tau}\tau}{MC_N}$ and $\varepsilon_A\equiv\frac{MC_{A\tau}\tau}{MC_A}$ where we use Eq.(22) in the text to eliminate n_t.

Plugging Eqs.(A.17)—(A.19) into Eq.(A.16) yields:

$$\nu=\kappa\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left\{\begin{aligned}&c_t+\varphi y_t+\frac{\tau}{1-\tau}\hat{\tau}_t-(1+\varphi)a_t+\sigma_C c_t^2+\varphi^2 y_t^2\\&+\frac{\tau\left[(1-\tau)(1-2\varepsilon_\tau)+\tau\right]}{2(1-\tau)^2}\hat{\tau}_t^2+\varphi(1+\sigma_C)c_t y_t\\&+\frac{(1-\tau)\varepsilon_C+\tau\sigma_C}{1-\tau}c_t\hat{\tau}_t-(1+\varphi)(1+\sigma_C)c_t a_t\\&+\sigma_G c_t g_t+\frac{\varphi\left[(1-\tau)\varepsilon_N+\tau\right]}{1-\tau}\hat{\tau}_t y_t-2\varphi y_t a_t+\sigma_G\varphi y_t g_t\\&-\frac{(1-\tau)(\varepsilon_A+\varphi)+\tau(1+\varphi)}{1-\tau}\hat{\tau}_t a_t+\frac{\sigma_G\tau}{1-\tau}\hat{\tau}_t g_t\end{aligned}\right\}+\text{s.o.t.i.p.}+o\left(\|\xi\|^3\right).$$

(A.20)

Second-order approximated period utility $U_t\equiv u(C_t)+v(N_t)$ with $u(C_t)\equiv\ln C_t$ and

$v(N_t)\equiv\frac{1}{1+\varphi}N_t^{1+\varphi}$ is given by:

$$u=\frac{\Phi}{\sigma_C}y_t-\left[\frac{1-\Phi}{\sigma_C}z_t+\frac{(1+\varphi)(1-\Phi)}{2\sigma_C}\left(y_t^2-2y_ta_t\right)\right]+\text{t.i.p.}+o\left(\|\xi\|^3\right),$$

with $u\equiv\frac{U(C_t,N_t)-U(C,N)}{U_cC}$, $\Phi\equiv1-\frac{v_N}{u_C}$ and $z_t\equiv\ln\left\{\int_0^1\left[\frac{P_t(i)}{P_t}\right]^{-\varepsilon}di\right\}$ where we use

$n_t=y_t+z_t+a_t$, $n_t^2=y_t^2+2y_ta_t-\text{s.o.t.i.p.}+o\left(\|\xi\|^3\right)$ and $\frac{N}{C}=\sigma_C^{-1}$.

Iterating the previous expression forward yields:

$$u=\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left\{\frac{\Phi}{\sigma_C}y_t-\left[\frac{(1-\Phi)\varepsilon}{2\kappa\sigma_C}\pi_t^2+\frac{(1+\varphi)(1-\Phi)}{2\sigma_C}\left(y_t^2-2y_ta_t\right)\right]\right\}+\text{t.i.p.}+o\left(\|\xi\|^3\right), \text{(A.21)}$$

where we use $\sum_{t=0}^{\infty}\beta^tz_t=\sum_{t=0}^{\infty}\beta^t\frac{\varepsilon}{2\kappa}\pi_t^2$.

In the first order, Eqs.(A.21), (A.14), (A.20) and (A.21) are given by:

$$u=\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left(\frac{\Phi}{\sigma_C}y_t\right)+\text{t.i.p.}+o\left(\|\xi\|^2\right), \text{(A.22)}$$

$$\varpi=(1-\delta-\beta)\sum_{t=0}^{\infty}\mathrm{E}_0\left[-c_t+\frac{(1+\omega_g)\omega_\phi}{1-\beta-\delta}\hat{\tau}_t+\frac{(1+\omega_g)\omega_\phi}{1-\beta-\delta}y_t\right]+\text{t.i.p}+\left(\|\xi\|^2\right), \text{(A.23)}$$

$$\nu=\kappa\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left\{c_t+\varphi y_t+\frac{\tau}{1-\tau}\hat{\tau}_t\right\}+\text{t.i.p.}+o\left(\|\xi\|^2\right), \text{(A.24)}$$

$$0=\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left(-y_t+\sigma_Cc_t\right)+\text{t.i.p}+o\left(\|\xi\|^2\right), \text{(A.25)}$$

where we ignore exogenous variables.

We formalize to eliminate y_t on Eq.(A.22) as follows:

$$\frac{\Phi}{\sigma_C}=\Theta_1\frac{(1+\omega_g)\omega_\phi}{1-\beta-\delta}+\Theta_2\varphi-\Theta_3,$$

$$0=\Theta_1\frac{(1+\omega_g)\omega_\phi}{1-\beta-\delta}+\Theta_2\frac{\tau}{1-\tau},$$

$$0=-\Theta_1+\Theta_2+\Theta_3\sigma_C,$$

where Θ_1, Θ_2 and Θ_3 are undetermined coefficients.

By solving the system, we have:

$$\Theta_1 = \frac{\Phi\tau(1-\beta-\delta)}{\Xi_0}, \qquad \Theta_2 = -\frac{\Phi(1-\tau)(1+\omega_g)\omega_\phi}{\Xi_0} \quad \text{and}$$

$$\Theta_3 = -\frac{\Phi\left[\tau(1-\beta-\delta)+(1-\tau)\omega_\phi(1+\omega_g)\right]}{\Xi_0\sigma_C},$$

with $\Xi_0 \equiv \omega_\phi(1+\omega_g)\left[\tau\sigma_C-(1-\tau)(1+\sigma_C\varphi)\right]-\tau(1-\beta-\delta)$.

$\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\frac{\Phi}{\sigma_C}y_t\right)$ on Eq.(A.22) can be expressed by using Eqs.(A.23), (A.24) and (A.25) as follows:

$$\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\frac{\Phi}{\sigma_C}y_t\right)=\Theta_1(1-\beta-\delta)^{-1}\varpi+\Theta_2\kappa^{-1}\nu$$

$$-\Theta_1\sum_{t=0}^{\infty}\mathrm{E}_0\left[\begin{array}{l}\frac{1}{2}c_t^2+\frac{(1+\omega_g)\omega_\phi}{2(1-\delta-\beta)}\hat{\tau}_t^2+\frac{(1+\omega_g)\omega_\phi}{2(1-\delta-\beta)}y_t^2\\ +\frac{\beta\left[1+(\gamma-1)^2\right]}{2(1-\beta-\delta)(\gamma-1)^2}\left(\hat{r}_t^R\right)^2-(1+\omega_g)c_t\hat{\tau}_t-(1+\omega_g)c_t y_t\\ +\omega_g c_t g_t+\frac{(1+\omega_g)\omega_\phi}{1-\beta-\delta}\hat{\tau}_t y_t\end{array}\right]$$

$$-\Theta_2\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left\{\begin{array}{l}\sigma_C c_t^2+\varphi^2 y_t^2+\frac{\tau\left[(1-\tau)(1-2\varepsilon_\tau)+\tau\right]}{2(1-\tau)^2}\hat{\tau}_t^2+\varphi(1+\sigma_C)c_t y_t\\ +\frac{(1-\tau)\varepsilon_C+\tau\sigma_C}{1-\tau}c_t\hat{\tau}_t-(1+\varphi)(1+\sigma_C)c_t a_t+\sigma_G c_t g_t\\ +\frac{\varphi\left[(1-\tau)\varepsilon_N+\tau\right]}{1-\tau}\hat{\tau}_t y_t-2\varphi y_t a_t+\sigma_G\varphi y_t g_t\\ -\frac{(1-\tau)(\varepsilon_A+\varphi)+\tau(1+\varphi)}{1-\tau}\hat{\tau}_t a_t+\frac{\sigma_G\tau}{1-\tau}\hat{\tau}_t g_t\end{array}\right\}$$

$$-\Theta_3\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left(\frac{\sigma_C\sigma_G}{2}c_t^2-\sigma_G c_t g_t\right)+\text{t.i.p}+o\left(\|\xi\|^3\right)$$

$$=-\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left\{\begin{array}{l}\left[\frac{\Theta_1(1+\omega_g)\omega_\phi}{2(1-\delta-\beta)}+\Theta_2\varphi^2\right]y_t^2+\left[\frac{\Theta_1}{2}+\Theta_2\sigma_C+\frac{\Theta_3\sigma_C\sigma_G}{2}\right]c_t^2\\ +\frac{\beta\Theta_1\left[1+(\gamma-1)^2\right]}{2(1-\beta-\delta)(\gamma-1)^2}\left(\hat{r}_t^R\right)^2\\ +\left[-\Theta_1(1+\omega_g)+\Theta_2\varphi(1+\sigma_C)\right]c_t y_t-2\Theta_2\varphi y_t a_t\\ +\left(\Theta_1\omega_g+\Theta_2\sigma_G-\sigma_G\Theta_3\right)c_t g_t-\Theta_2(1+\varphi)(1+\sigma_C)c_t a_t\\ +\Theta_2\sigma_G\varphi y_t g_t\end{array}\right\},$$

$$-\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\mathrm{T}_t+\Upsilon_0+\text{t.i.p.}+o\left(\|\xi\|^3\right)$$

with:

$$
\begin{aligned}
\mathrm{T}_t \equiv & \left[\frac{\Theta_1(1+\omega_g)\omega_\phi}{1-\beta-\delta}+\frac{\Theta_2\varphi\left[(1-\tau)\varepsilon_N+\tau\right]}{1-\tau}\right]\hat{\tau}_t y_t \\
& +\left\{-\Theta_1(1+\omega_g)+\frac{\Theta_2\left[(1-\tau)\varepsilon_C+\tau\sigma_C\right]}{1-\tau}\right\}c_t\hat{\tau}_t \\
& -\frac{\Theta_2\left[(1-\tau)(\varepsilon_A+\varphi)+\tau(1+\varphi)\right]}{1-\tau}\hat{\tau}_t a_t+\frac{\Theta_2\sigma_G\tau}{1-\tau}\hat{\tau}_t g_t+\frac{\Theta_2\tau\left[(1-\tau)(1-2\varepsilon_\tau)+\tau\right]}{2(1-\tau)^2}\hat{\tau}_t^2
\end{aligned},
$$

$$
\Upsilon_0 \equiv \Theta_1(1-\beta-\delta)^{-1}\varpi+\Theta_2\kappa^{-1}\nu .
$$

We set $\varepsilon_N=-\dfrac{\Theta_1(1-\tau)(1+\omega_g)\omega_\phi+\Theta_2\tau\varphi(1-\beta-\delta)}{\Theta_2\varphi(1-\beta-\delta)(1-\tau)}$, $\varepsilon_C=-\dfrac{\Theta_1(1+\omega_g)}{\Theta_2}$, $\varepsilon_\tau\equiv-\dfrac{1}{2(1-\tau)}$ and $\varepsilon_A=-\dfrac{\tau+\varphi}{1-\tau}$. Then, $\mathrm{T}_t=0$ so that:

$$
\begin{aligned}
\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\frac{\Phi}{\sigma_C}y_t\right)= & -\sum_{t=0}^{\infty}\mathrm{E}_0\left[\tilde{\Omega}_1 y_t^2-2\left(\Omega_2 y_t g_t+\tilde{\Omega}_3 y_t a_t\right)+\frac{\beta\left[1+(\gamma-1)^2\right]}{2(1-\beta-\delta)(\gamma-1)^2}\left(\hat{r}_t^R\right)^2\right], \\
& +\Upsilon_0++o\left(\|\xi\|^3\right)
\end{aligned}
$$

with:

$$
\tilde{\Omega}_1 \equiv \frac{\Theta_1(1+\omega_g)\sigma_C^2\omega_\phi+(1-\beta-\delta)\left\{\begin{array}{l}\Theta_1(1-2\sigma_C)+\Theta_3\sigma_G\sigma_C \\ +2\sigma_C\left[\Theta_2\left(\varphi\left(\sigma_C(1+\varphi)+1\right)+1\right)-\Theta_1\omega_g\right]\end{array}\right\}}{2\sigma_C^2(1-\beta-\delta)},
$$

$$
\Omega_2 \equiv \frac{\Theta_1\left\{\sigma_G-\sigma_C\left[(1+\omega_g)\sigma_G+\omega_g\right]\right\}+\sigma_C\sigma_G\left[\Theta_2(1+\varphi)+\Theta_3(1+\sigma_G)\right]}{2\sigma_C^2},
$$

$$
\tilde{\Omega}_3 \equiv \frac{\Theta_2\left[2\sigma_C\varphi+(1+\varphi)(1+\sigma_C)\right]}{2\sigma_C}.
$$

Plugging the previous expression into Eq.(A.21) yields:

$$
\begin{aligned}
u=-\sum_{t=0}^{\infty}\mathrm{E}_0 & \left[\Omega_1 y_t^2-2(\Omega_2 y_t g_t+\Omega_3 y_t a_t)+\frac{(1-\Phi)\varepsilon}{2\kappa\sigma_C}\pi_t^2+\frac{\beta\Theta_1\left[1+(\gamma-1)^2\right]}{2(1-\beta-\delta)(\gamma-1)^2}\left(\hat{r}_t^R\right)^2\right], \\
& +\Upsilon_0++o\left(\|\xi\|^3\right)
\end{aligned}
$$

with $\Omega_1 \equiv \tilde{\Omega}_1 + \frac{(1+\varphi)(1-\Phi)}{2\sigma_C}$ and $\Omega_3 \equiv \tilde{\Omega}_3 + \frac{(1+\varphi)(1-\Phi)}{2\sigma_C}$, which can be rewritten as:

$$u = -\sum_{t=0}^{\infty} \mathrm{E}_0 \left[\Omega_1 \left(y_t - y_t^* \right) + \frac{(1-\Phi)\varepsilon}{2\kappa\sigma_C} \pi_t^2 + \frac{\beta\Theta_1 \left[1 + (\gamma-1)^2 \right]}{2(1-\beta-\delta)(\gamma-1)^2} \left(\hat{r}_t^R \right)^2 \right] + \Upsilon_0 + + o\left(\|\xi\|^3 \right),$$

with $y_t^* \equiv \frac{\Omega_2}{\Omega_1} g_t + \frac{\Omega_3}{\Omega_1} a_t$, which is Eq.(31) in the text. Let define $\Lambda_y \equiv 2\Omega_1$, $\Lambda_\pi \equiv \frac{(1-\Phi)\varepsilon}{\kappa\sigma_C}$ and $\Lambda_r \equiv \frac{\beta\Theta_1 \left[1 + (\gamma-1)^2 \right]}{(1-\beta-\delta)(\gamma-1)^2}$. Then, the previous expression can be rewritten as:

$$u = -\sum_{t=0}^{\infty} \mathrm{E}_0 \left(L_t \right) + \Upsilon_0 + + o\left(\|\xi\|^3 \right),$$

with:

$$L_t \equiv \frac{\Lambda_y}{2} \left(y_t - y_t^* \right)^2 + \frac{\Lambda_\pi}{2} \pi_t^2 + \frac{\Lambda_r}{2} \left(\hat{r}_t^G - \hat{r}_t \right)^2,$$

which is Eq.(30) in the text.

A.2 Derivation of Second-order approximated utility function without Default Risk

Let define:

$$W_t^f \equiv \sum_{t=0}^{\infty} \beta^t \mathrm{E}_0 C_t^{-1} SP_t \text{ , (A.26)}$$

with $W_t^f \equiv C_t^{-1} R_{t-1} \Gamma\left(-sp_t \right) B_{t-1} \Pi_t^{-1}$. (A.27)

Second-order approximation of Eq.(A.26) is given by:

$$w_t^f = (1-\beta) \left(-c_t + sp_t + \frac{1}{2} c_t^2 - c_t sp_t \right) + \beta \mathrm{E}_t \left(w_{t+1}^f \right).$$

Iterating the previous expression yields:

$$\varpi^f = (1-\beta) \sum_{t=0}^{\infty} \beta^t \left(-c_t + sp_t + \frac{1}{2} c_t^2 - c_t sp_t \right) + o\left(\|\xi\|^3 \right)$$

with $\varpi^f \equiv w_0$.

Plugging Eq.(A.10) into the previous expression yields:

$$\varpi^f = (1-\beta)\sum_{t=0}^{\infty}\beta^t \begin{bmatrix} -c_t + (1+\omega_g)\hat{\tau}_t + (1+\omega_g)y_t + \frac{1}{2}c_t^2 \\ +\frac{1+\omega_g}{2}\hat{\tau}_t^2 + \frac{1+\omega_g}{2}\hat{y}_t^2 - (1+\omega_g)c_t\hat{\tau}_t \\ -(1+\omega_g)c_t y_t + \omega_g c_t g_t + (1+\omega_g)\hat{\tau}_t y_t \end{bmatrix} + o\left(\|\xi\|^3\right), \text{ (A.28)}$$

In the first order, Eq.(A.28) is given by:

$$\varpi^f = (1-\beta)\sum_{t=0}^{\infty}\beta^t \left[-c_t + (1+\omega_g)\hat{\tau}_t + (1+\omega_g)y_t + \frac{1}{2}c_t^2 \right] + o\left(\|\xi\|^3\right). \text{ (A.29)}$$

We formalize to eliminate y_t on Eq.(A.22) as follows:

$$\frac{\Phi}{\sigma_C} = \Theta_1^f\left(1+\omega_g\right) + \Theta_2^f\varphi - \Theta_3^f, \text{ (A.30)}$$

$$0 = \Theta_1^f\left(1+\omega_g\right) + \Theta_2^f\frac{\tau}{1-\tau}, \text{ (A.31)}$$

$$0 = -\Theta_1^f\left(1+\omega_g\right) + \Theta_2^f + \Theta_3^f\sigma_C, \text{ (A.32)}$$

where Θ_1^f, Θ_2^f and Θ_3^f are undetermined coefficients on Eqs.(A.29), (A.24) and (A.25), respectively. Solution of the previous system yields:

$$\Theta_1^f = \frac{\Phi\tau}{\Xi_0^f},$$

$$\Theta_2^f = -\frac{\Phi(1-\tau)\left(1+\omega_g\right)}{\Xi_0^f}$$

$$\Theta_3^f = -\frac{\Phi\left[\tau + (1-\tau)\left(1+\omega_g\right)\right]}{\Xi_0^f}$$

with $\Xi_0^f \equiv \left(1+\omega_g\right)\left[\tau\sigma_C - (1-\tau)\left(1+\sigma_C\varphi\right)\right] - \tau$.

$\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\frac{\Phi}{\sigma_C}y_t\right)$ on Eq.(A.22) can be expressed by using Eqs.(A.29), (A.24) and (A.25) as follows:

$$\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\frac{\Phi}{\sigma_C}y_t\right)=\Theta_1^f\left(1-\beta-\delta\right)^{-1}\varpi^f+\Theta_2^f\kappa^{-1}\nu$$

$$-\Theta_1^f\sum_{t=0}^{\infty}\mathrm{E}_0\left[\begin{array}{l}\frac{1}{2}c_t^2+\frac{1+\omega_g}{2}\hat{\tau}_t^2+\frac{1+\omega_g}{2}y_t^2-\left(1+\omega_g\right)c_t\hat{\tau}_t-\left(1+\omega_g\right)c_t y_t\\ +\omega_g c_t g_t\left(1+\omega_g\right)\hat{\tau}_t y_t\end{array}\right]$$

$$-\Theta_2^f\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left\{\begin{array}{l}\sigma_C c_t^2+\varphi^2 y_t^2+\frac{\tau\left[(1-\tau)(1+2\varepsilon_\tau)+\tau\right]}{2(1-\tau)^2}\hat{\tau}_t^2+\varphi\left(1+\sigma_C\right)c_t y_t\\ +\frac{(1-\tau)\varepsilon_C+\tau\sigma_C}{1-\tau}c_t\hat{\tau}_t-(1+\varphi)(1+\sigma_C)c_t a_t+\sigma_G c_t g_t\\ +\frac{\varphi\left[(1-\tau)\varepsilon_N+\tau\right]}{1-\tau}\hat{\tau}_t y_t-2\varphi y_t a_t+\sigma_G\varphi y_t g_t\\ -\frac{(1-\tau)(\varepsilon_A+\varphi)+\tau(1+\varphi)}{1-\tau}\hat{\tau}_t a_t+\frac{\sigma_G\tau}{1-\tau}\hat{\tau}_t g_t\end{array}\right\}$$

$$-\Theta_3^f\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left(\frac{\sigma_C\sigma_G}{2}c_t^2-\sigma_G c_t g_t\right)+\text{t.i.p}+o\left(\left\|\xi\right\|^3\right)$$

$$=-\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\left\{\begin{array}{l}\left[\frac{\Theta_1^f\left(1+\omega_g\right)}{2}+\Theta_2^f\varphi^2\right]y_t^2+\left[\frac{\Theta_1^f}{2}+\Theta_2^f\sigma_C+\frac{\Theta_3^f\sigma_C\sigma_G}{2}\right]c_t^2\\ +\left[-\Theta_1^f\left(1+\omega_g\right)+\Theta_2^f\varphi\left(1+\sigma_C\right)\right]y_t c_t-2\Theta_2\varphi y_t a_t\\ +\left(\Theta_1^f\omega_g+\Theta_2^f\sigma_G-\sigma_G\Theta_3^f\right)c_t g_t-\Theta_2^f\left(1+\varphi\right)\left(1+\sigma_C\right)c_t a_t\\ +\Theta_2^f\sigma_G\varphi y_t g_t\end{array}\right\},$$

$$-\sum_{t=0}^{\infty}\beta^t\mathrm{E}_0\mathrm{T}_t^f+\Upsilon_0^f+\text{t.i.p.}+o\left(\left\|\xi\right\|^3\right)$$

with:

$$\mathrm{T}_t^f\equiv\left[\Theta_1^f\left(1+\omega_g\right)+\frac{\Theta_2^f\varphi\left[(1-\tau)\varepsilon_N^f+\tau\right]}{1-\tau}\right]\hat{\tau}_t y_t+\left\{-\Theta_1^f\left(1+\omega_g\right)+\frac{\Theta_2^f\left[(1-\tau)\varepsilon_C^f+\tau\sigma_C\right]}{1-\tau}\right\}c_t\hat{\tau}_t$$

$$-\frac{\Theta_2^f\left[(1-\tau)\left(\varepsilon_A^f+\varphi\right)+\tau(1+\varphi)\right]}{1-\tau}\hat{\tau}_t a_t+\frac{\Theta_2\sigma_G\tau}{1-\tau}\hat{\tau}_t g_t+\frac{\Theta_2^f\tau\left[(1-\tau)\left(1+2\varepsilon_\tau^f\right)+\tau\right]}{2(1-\tau)^2}\hat{\tau}_t^2$$

, $\Upsilon_0^f\equiv\Theta_1^f\left(1-\beta-\delta\right)^{-1}\varpi^f+\Theta_2^f\kappa^{-1}\nu$.

We set $\varepsilon_N^f=-\frac{\Theta_1^f(1-\tau)\left(1+\omega_g\right)+\Theta_2^f\tau\varphi}{\Theta_2^f\varphi(1-\tau)}$, $\varepsilon_C^f=-\frac{\Theta_1^f\left(1+\omega_g\right)}{\Theta_2^f}$, $\varepsilon_\tau^f=-\frac{1}{2(1-\tau)}$ and

$\varepsilon_A^f = -\frac{\tau+\varphi}{1-\tau}$. Then, $\Upsilon_t^f = 0$ so that:

$$\sum_{t=0}^{\infty}\beta^t \mathrm{E}_0\left(\frac{\Phi}{\sigma_C}y_t\right) = -\sum_{t=0}^{\infty}\mathrm{E}_0\left[\tilde{\Omega}_1^f y_t^2 - 2\left(\Omega_2^f y_t g_t + \tilde{\Omega}_3^f y_t a_t\right)\right] + \Upsilon_0^f + +o\left(\|\xi\|^3\right),$$

with:

$$\tilde{\Omega}_1^f \equiv \frac{\Theta_1^f\left[\left(1+\omega_g\right)\sigma_C^2 + \left(1-2\sigma_C\right)\right] + \Theta_3^f\sigma_G\sigma_C + 2\sigma_C\left[\Theta_2^f\left(\varphi\left(\sigma_C\left(1+\varphi\right)+1\right)+1\right) - \Theta_1^f\omega_g\right]}{2\sigma_C^2},$$

$$\Omega_2^f \equiv \frac{\Theta_2^f\left\{\sigma_G - \sigma_C\left[\left(1+\omega_g\right)\sigma_G + \omega_g\right]\right\} + \sigma_C\sigma_G\left[\Theta_2^f\left(1+\varphi\right) + \Theta_3^f\left(1+\sigma_G\right)\right]}{2\sigma_C^2},$$

$$\tilde{\Omega}_3^f \equiv \frac{\Theta_2^f\left[2\sigma_C\varphi + \left(1+\varphi\right)\left(1+\sigma_C\right)\right]}{2\sigma_C}.$$

Plugging the previous expression into Eq.(A.21) yields:

$$u = -\sum_{t=0}^{\infty}\mathrm{E}_0\left[\Omega_1^f y_t^2 - 2\left(\Omega_2^f y_t g_t + \Omega_3^f y_t a_t\right) + \frac{\left(1-\Phi\right)\varepsilon}{2\kappa\sigma_C}\pi_t^2\right] + \Upsilon_0^f + +o\left(\|\xi\|^3\right),$$

with $\Omega_1^f \equiv \tilde{\Omega}_1^f + \frac{\left(1+\varphi\right)\left(1-\Phi\right)}{2\sigma_C}$ and $\Omega_3^f \equiv \tilde{\Omega}_3^f + \frac{\left(1+\varphi\right)\left(1-\Phi\right)}{2\sigma_C}$, which can be rewritten as:

$$u = -\sum_{t=0}^{\infty}\mathrm{E}_0\left[\Omega_1^f\left(y_t - y_t^f\right) + \frac{\left(1-\Phi\right)\varepsilon}{2\kappa\sigma_C}\pi_t^2\right] + \Upsilon_0 + +o\left(\|\xi\|^3\right),$$

with $y_t^f \equiv \frac{\Omega_2^f}{\Omega_1^f}g_t + \frac{\Omega_3^f}{\Omega_1^f}a_t$. Let define $\Lambda_y^f \equiv 2\Omega_1^f$. Then, the previous expression can be rewritten as:

$$u = -\sum_{t=0}^{\infty}\mathrm{E}_0\left(L_t^f\right) + \Upsilon_0^f + +o\left(\|\xi\|^3\right),$$

with:

$$L_t \equiv \frac{\Lambda_y^f}{2}\left(y_t - y_t^f\right)^2 + \frac{\Lambda_\pi}{2}\pi_t^2,$$

which is Eq.(32) in the text.